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EXECUTIVE OPTIONS: VALUATION AND PROJECTION METHODOLOGIES

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ABSTRACT

The valuation of options awarded to senior executives of listed companies is a high profile and contentious issue.

The following paper is concerned with the valuation of executive options - particularly those involving performance hurdles. Given the complexity of the underlying benefit designs, valuation of these options will frequently require simulation techniques.

Accordingly the paper sets out methodologies for these techniques, within the risk-neutral framework commonly used for pricing options on shares.

A methodology for estimating probabilities of achieving performance hurdles is also put forward.

KEYWORDS

Executive options, Performance hurdles, Simulation methods, Risk-neutral valuation

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1. ACKNOWLEDGEMENTS

Work of this nature relies on the goodwill of many parties. The authors have extended the patience of many along the way – and would like to take the opportunity to profusely apologise to all of these persons.

The authors would like to sincerely thank their former employers, Tillinghast-Towers Perrin and Towers Perrin, and in particular David Solomon and Janet Linklater, for their permission to publish this work.

Readers of this paper should note that:

1. The work was originally developed by Towers Perrin and Tillinghast-Towers Perrin to address their perception that insufficient rigour was (generally) being applied to the valuation of executive options with performance hurdles.
2. Towers Perrin is currently doing work for clients in this area, based on the original work presented in this paper.
3. Towers Perrin's valuation methodology has been further developed, by various additions to this basic approach.
4. Towers Perrin is continuing to research and develop valuation tools for these types of options.

Tim Kyng of the IAAust Banking and Finance Practice committee provided useful peer review and his suggestions have helped us to improve the paper. All remaining errors are of course our own.

A special thanks must also go to Tig Melville, whose generous sponsorship of the Melville Practitioners' Prize (in honour of his father Sir Leslie Melville) helped to facilitate the publication of this work. Naturally, the views expressed in this paper are those of the authors. These views are not necessarily held by their former or current employers.

2. PROLOGUE

'……..like hapless water buffaloes stuck in the mud waiting for the safari to arrive with the elephant gun to put them out of their misery. Hopefully we pulled up in the ute within shooting distance just in time.' (H.G. Nelson 1996).

Although Mr Nelson was not referring to executive option valuation, the 'hapless water buffalo' is an apt metaphor for the current debate on this issue. Recent times have seen an intense focus on executive compensation, in particular, options. While this focus appears to have led to significant gains in disclosure and transparency, the same cannot be said for the valuation of executive options. In practice, one observes a wide diversity of views and practices, even between companies with similar plans. Of more concern is the view held by some that option plans, especially those with performance hurdles, are too complex to value properly. This seems inevitably to lead to one of two unsavoury practices, i.e. the use of methods that are technically unsound, or worse - to the avoidance of calculating a value at all!

Hence the motivation for this paper. What we aim to demonstrate is that even very complex executive option plans can be valued on a sound technical basis. Further, to do so does not require much more financial calculus than most actuaries (at least nowadays) will learn on their way to qualification. Our further aim, then, is to add to the actuarial 'toolkit' required for building executive option models. Fully kitted out, so to speak, the actuarial profession has an immense opportunity to perform valuable work in this area.

It is well past time that the 'hapless water buffalo' of executive option valuation was mercifully put to rest. We hope that this paper helps us all towards this worthwhile goal.

2.1 Why Valuation Matters

While current valuation practice may be unsatisfactory, this is only of concern if sound valuation practice is considered desirable. Ultimately, proper valuations of executive options and projections of probable outcomes are in the interests of shareholders and executives. More robust methods will provide better information with respect to:

- Benchmarking executive remuneration packages against both absolute and relative (to other companies) measures.
- Determining the number of options to be granted to the executive where a fixed dollar value of options has been allocated.
- Ascertaining the quantum of wealth being transferred from shareholders to executives.
- Ascertaining the level of difficulty associated with attaining complex performance hurdles, and the corresponding probabilities attaching to various levels of executive reward (and their relativity to the quantum of shareholder reward in such situations).

3. INTRODUCTION

3.1 Structure of this paper

The structure of this paper is as follows:

- This section explains why simulation methods will frequently be required for valuing executive options.
- Section 4 outlines the primary tools and algorithms required for executive option valuation.
- Sections 5 to 8 provide examples of how these techniques can be applied to 4 specific types of executive option.
- Section 9 provides a worked example of our simulation methodology, along with some sample values for the 4 types of executive option plans considered.
- Section 10 discusses the setting of assumptions.
- Section 11 outlines some of the limitations of our approach.
- Detailed discussion of the theoretical and practical considerations underlying our suggested methodology can be found in the appendices.

3.2 Black-Scholes, performance hurdles and simulation methods

When Fisher Black and Myron Scholes published their work on the valuation of call and put options in 1973, they provided a framework that can be used to value all manner of derivatives – including executive options. The famous Black-Scholes formula for call options, however, relates to standard European call options. These options involve the right (but not the obligation) to purchase a security for a fixed price at some fixed date in the future.

While they possess call option features, virtually all executive option plans will be more complex than a vanilla European call option. This is primarily due to the fact that most executive option plans involve performance hurdles. Performance hurdles have two important and related effects:

- They reduce the probability of the option being exercised (relative to the case in which there is no performance hurdle); and
- Valuation becomes considerably more complex, rendering the standard Black-Scholes call option formula insufficient for valuation purposes.

In practice, though, one frequently sees executive option valuation performed using 'rough and ready' approaches such as:

- The estimated 'real world' (as opposed to the Black-Scholes consistent risk-neutral world – the probability measures in each are different) probability of attaining the performance hurdle; times
- The Black-Scholes value of the option without the performance hurdle.

As discussed in Carrett, Miller and Roberts (2000), this method is highly unsatisfactory (ie wrong). This article points out that even where the real world probability of attaining a performance hurdle is 50% (or thereabouts) the true value of the option may well be of the order of 80% to 90% of this Black-Scholes value. Real world probabilities and risk-neutral Black-Scholes analysis do not mix.

In this paper we put forward a valuation framework that uses the Black-Scholes risk-neutral framework to value executive options allowing for the existence of performance hurdles. That is, we apply the very framework used by Black and Scholes to value standard European calls and puts to the problem of executive option valuation. This leads to a valuation framework that (we feel) is theoretically sound, readily capable of implementation and defensible to users of this information.

A further (though distinct) problem of interest to those working in this area – not to mention executives – is the estimation of the real world probability of performance hurdles being achieved, and the related frequency distribution of dollar payoffs to the executive. We also investigate this problem.

We have chosen to discuss these valuation and probability issues via four sample executive option designs, each with a different performance hurdle. The performance hurdles that we have chosen include:

- Fixed share price hurdles; and
- Three different types of options involving relative Total Shareholder Return (TSR) hurdles, ie. where:
    - the TSR is measured at a single point in time relative to a given index;
    - the TSR is measured against an index, but where there is a period of time over which the hurdle criteria can be met; and
    - the TSR performance is measured relative to a peer group of stocks (across several currencies).

Before we discuss each of these cases, however, it is useful to contemplate a broad framework for valuing executive options using simulation techniques.

4. EXECUTIVE OPTIONS: A GENERAL FRAMEWORK

Executive option packages typically have a structure resembling the following:

- The executive is allocated a number of options.
- The number of options ultimately vesting in the executive will vary according to some measure of company performance, such as TSR or share price growth.
- Often this performance measure will be relative to one of:
  - a sharemarket index;
  - a peer group of stocks; or
  - a fixed share price hurdle.
- This measurement of company performance will take place over a period determined at the outset. This period is typically three to five years. Hereafter we refer to this period as the measurement period.
- Once vested in the executive, the options are typically exercisable immediately (ie. American options), although the final possible exercise date is often several years after vesting.
- The exercise or strike price of the options is based on some average market price (say, over the preceding 5 trading days) prior to the initial allocation of options.

This structure is summarised in Figure 1.

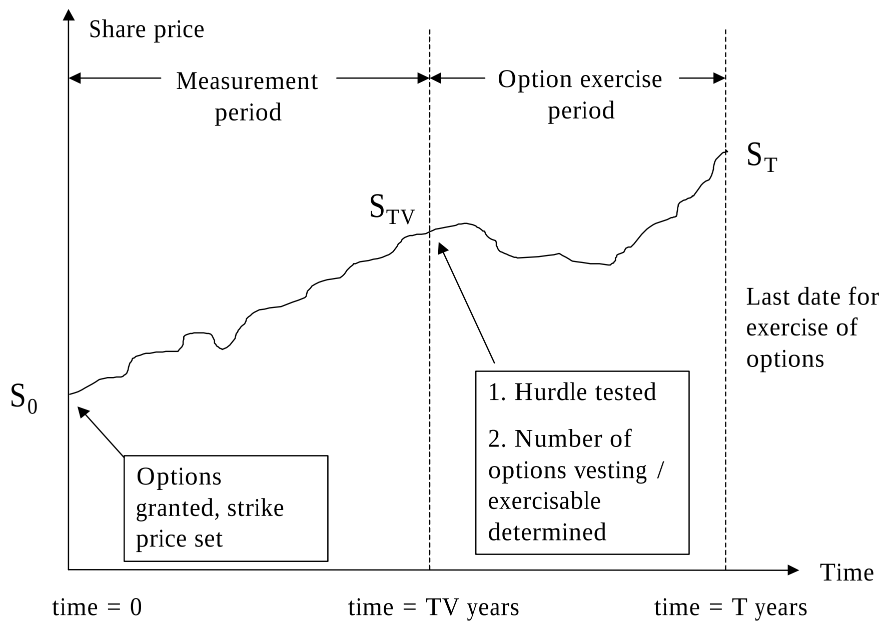

Figure 1: Typical Structure (S_t = share price at time t)

Given the complexity of these option packages, closed-form (ie. formula driven) valuation solutions are generally not possible, meaning that simulation methods will be required. These complications generally relate to the complex TSR hurdles embedded in executive options.

4.1 Outline of simulation methodology

(a) Simulation methods – Option Valuation

Simulation methods involve projecting future outcomes based on the statistical distributions of key random variables. In the simulation models that form the basis of this paper, samples are taken from the distributions assumed to apply to various stock and index return measures. Using these samples, the models then project the value that would be assigned in each individual simulation to the executive option. By performing enough simulations, we can take an average option value across all simulations to obtain an estimate of the true option value.

More specifically, a typical algorithm for valuing an executive option in this way works as follows:

- Stock and Index TSR measures (or some other return measure) are randomly generated over the measurement period. These returns are sampled from an appropriate statistical distribution.
- TSR comparisons (or whichever performance hurdle applies) between the stocks and/or index are used to determine:
    - Whether or not the option vests in the executive; and/or
    - How many options vest in the executive.
- The share price at the end of the performance measurement period is then derived (as implied by the randomly generated return). This share price is used to value each option vested in the executive. The Black-Scholes equation would almost invariably be used.
- This process would be repeated many thousands of times. The average value of the option across all simulations gives us our estimate of the option value.

The structure of such algorithms is represented diagrammatically in Figure 2.

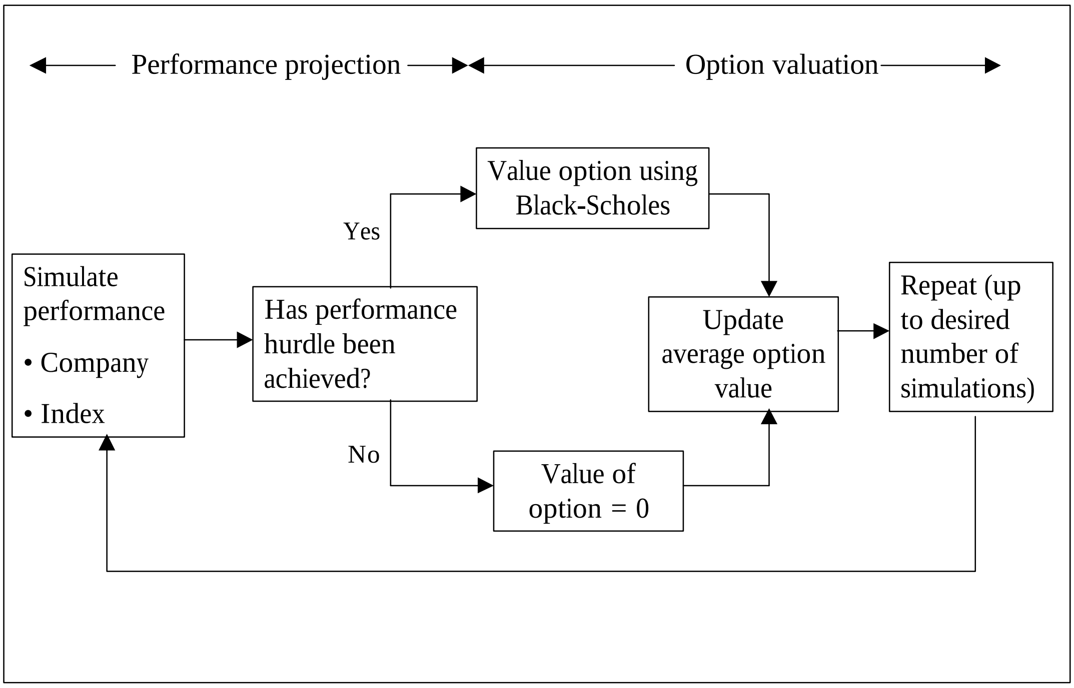

Figure 2: Simulation Model – Outline (Source: Carrett, Miller and Roberts (2000))

(b) Projection of returns

From the above it can be seen that the projection of returns is central to simulation methodologies. For options, valuation methods rely on the principles of risk-neutral valuation (see Appendix A). In the related valuation models, returns are projected using:

- A risk-neutral expected return. This is the continuous time risk-free rate r, adjusted for dividends at rate q in the case of share price growth $(r – q – \sigma^2/2)$. Adding back dividend return when calculating TSR gives $(r - q + q – \sigma^2/2) = r – \sigma^2/2$.
- A random error term to allow for the expected volatility of returns. These random error terms make further allowance as required for the correlation between the returns of different securities. This is achieved using the Cholesky decomposition (refer Appendix C) to determine factors that, when applied to the random errors – initially independent – produce random errors and hence randomly generated returns that have not only the required standard deviation but also the required correlation structure.

The mechanics for performing such return projections are set out below.

Table 1: Option Valuation Variables

Variable	Description
S_t	Stock price at time t
r	Risk free interest rate (continuous) per annum
σ_i	Volatility of per annum returns for stock i
q_i	Dividend yield per annum for stock i, also continuous.
α_{ij}	Cholesky factors for correlation structure
t	Period over which we are projecting returns
e_i	Random N(0,1) variable.
n	Number of stocks being projected.

Generally it is necessary to project one or more of:

- Total shareholder returns; and
- Stock price growth.

The randomly simulated TSR for stock i over a period of length t, denoted as r_i^* is then given by[i]:

$r_i^* = (r - \sigma_i^2/2)t + \sigma_i\, e_i \sqrt{t}$.

where the e_i are first randomly generated as a series of independent N(0,1) variables subsequently transformed to be dependent random variables via methods such as the Cholesky decomposition.

The continuous rate of growth in the stock price over the period, adjusting for dividends, is then given by $r_i^* - q_i\, t$. That is, if the initial stock price is S_0 then the stock price will grow to $S_0.\exp\{r_i^* - q_i\, t\}$ at time t.

(c) Generation of random variables

The above discussion glosses over the vexing issue of generating random variables, in this instance random unit normal variables. Our preferred approach relies on:

- First, producing random numbers distributed evenly across the interval (0,1); and
- Converting these U(0,1) random variables to N(0,1) random variables.

The first of these tasks can be admirably achieved using the PMBD method outlined in Press et al (1995), and the second using an algorithm contained in Moro (1985). Refer to Appendix E for further elaboration of this issue.

(d) Number of simulations

With any simulation routine one needs a criterion for determining how many simulations are required. In essense, this criterion should have regard to the number of simulations required for the average (of the function being investigated) across all simulations to stabilise. Hence one could look at statistics such as the variance of simulation results and approximate confidence intervals. Detailed guidance on these issues can be found in Hossack, Pollard and Zehnwirth (1983), as well as Clewlow and Strickland (1998).

It should also be noted that the efficiency of simulation algorithms can be significantly improved by employing variance reduction techniques (see Appendix F). The authors strongly encourage practitioners to use such methods where practical.

(e) CAPM projections

The above discussion relates to the projection of returns in the risk-neutral world. Often, however, the modeller may wish to have a guide as to how likely a hurdle is to be attained in the 'real world'. The methodology that we propose for estimating such real world probabilities of attaining hurdles is set out below.

In short, our method involves:

- A continuous time version of the Capital Asset Pricing Model (CAPM). That is, rather than using an annual single-period CAPM rate of return as is often employed, we use the continuous form of the model (ie. δ instead of i, where i is an annual in arrears interest rate and where $\delta = \log_e(1 + i)$).
- An assumption that no serial correlation exists between time periods (although correlation between the returns of the various stocks and indices is allowed for). Hence our model is a form of the random walk model for stock prices and TSR. Combined with the assumption of a Gaussian return distribution this implies the assumption that returns over different time periods are independent.

Note that the CAPM rate of return is, by definition, TSR oriented.

The returns generated by this model follow a similar structure to that used for the risk-neutral world. That is, the randomly generated returns are equal to:

- An expected return equal to the CAPM return assumed for the stock/index; plus
- A random error term to allow for the expected volatilities of returns. The volatility measure is the same in the risk-neutral world as it is in the CAPM world. As per the risk-neutral world, these random error terms make allowances for expected correlation between the returns of different securities.

Accordingly, the process of generating 'real world' returns is similar to the process used for risk-neutral return generation.

The simulated TSR return for security i over a period of length t, R_i^* is then given by:

$$R_i^* = (\mu_i - \sigma_i^2/2)t + \sigma_i e_i \sqrt{t}.$$

Here μ_i is the instantaneous CAPM expected rate of return, calculated using the formula:

$\mu_i = r_f + \beta_i (r_m - r_f)$

where:

- r_f is the risk-free rate;
- β_i is the beta of stock (or index) i; and
- $(r_m - r_f)$ is the equity risk premium.

Note that as we are assuming a continuous CAPM rate of return, these parameters should relate to continuous returns. The e_i are generated using the process described above for risk-neutral valuation.

The continuous rate of growth in the stock price over the period is then given by R_i^* - q_i t. That is, if the initial stock price is S_0 then the stock price will grow to $S_0.\exp\{R_i^* - q_i\ t\}$ at time t.

Armed with the above algorithms, we are now ready to tackle some executive option problems.

5. OPTION TYPE 1: FIXED SHARE PRICE HURDLE

5.1 Description of option

These options are among the simplest of executive options. Though commonly found in the US, options plans with performance hurdles based on a fixed share price target (generally at some level above the current share price) are not often employed in Australia.

Table 2: A sample design of Option Type 1

Parameter	Symbol	Description/Comment
Hurdle / Target	H	Share price must be above a fixed level H at the end of the measurement period.
Measurement period		The hurdle is tested exactly 3 years after the date of allocation.
Strike price	X	Share price on the day of the initial allocation.
Vesting Period	TV	The time from the valuation date until the end of the measurement period, as the hurdle is only tested once.
Period until Exercise	T	Option exercisable at the expiry of 5 years from the initial allocation date.

The vested option's payoff is equal to the difference between the share price and the exercise price at the exercise date (subject to a minimum of zero).

5.2 Valuation methodology

This type of option can be valued using a closed-form formula. The relevant formula can be derived using risk-neutral valuation, and is set out below.

$$\text{Value} = S_0 \,.\, e^{-qT} \,.\, N_2(a_1,b_1,g) - X \,.\, e^{-rT} \,.\, N_2(a_2,b_2,g)$$

Where

$a_1 = [\log(S_0 / X) + (r - q + \sigma^2/2)\, T\,] / (\sigma \,.\, T^{0.5})$,

$b_1 = [\log(S_0 / H) + (r - q + \sigma^2/2)\, TV\,] / (\sigma \,.\, TV^{0.5})$,

$a_2 = [\log(S_0 / X) + (r - q - \sigma^2/2)\, T\,] / (\sigma \,.\, T^{0.5}) = a_1 - \sigma \,.\, T^{0.5}$,

$b_2 = [\log(S_0 / H) + (r - q - \sigma^2/2)\, TV\,] / (\sigma \,.\, TV^{0.5}) = b_1 - \sigma \,.\, TV^{0.5}$,

$g = (TV / T)^{0.5}$,

TV = Period from allocation date until vesting date,

H = Share price hurdle,

$N_2(x,y,z)$ = The cumulative bivariate normal distribution representing the joint probability {P(X<x) and P(Y<y)}, where z = correlation between X and Y. Appendix D refers the interested reader to a numerical procedure for calculating this probability.

5.3 Real world projection methodology

Similarly, the continuous CAPM probability of hitting the hurdle can be calculated without resorting to simulation methods.

$$\text{Prob} = \Pr(S_{TV} > H)$$
$$= \Pr(Z < [\ln(S_0 / H) + (\mu - q - \sigma^2/2)\, TV\,] / [\sigma \,.\, TV^{0.5}\,])$$
$$= \Pr(Z < b_2 + [(\mu - r)TV)] / [\sigma \,.\, TV^{0.5}\,]),$$

where $Z \sim N(0,1)$ and μ is the TSR expected for the stock based on the continuous time CAPM model.

6. OPTION TYPE 2: RELATIVE TSR HURDLE

6.1 Description of option

Like the fixed share price hurdle of the previous section, the relative TSR hurdle described in this section is tested on a particular date.

Table 3: A sample design of Option 2

Parameter	Symbol	Description
Hurdle / Target		Cumulative Company TSR must outperform the given Index's TSR.
Measurement period		The hurdle is tested exactly 3 years from the date of allocation (TV).
Strike price	X	Average trading price over the 5 trading days preceding initial allocation.
Vesting Period	TV	The time from the valuation date until the end of the measurement period.
Period until Exercise	T	Option exercisable at the expiry of 5 years from the initial allocation date.

The hurdle of these options will almost invariably require the company's TSR to be greater than the TSR of either a broad market index or some sub-index of the stockmarket.

6.2 Valuation methodology

This type of option can also be valued by means of a closed-end formula using risk-neutral valuation.

The value of the option can be calculated using the formula:

$$\text{Value} = S_0 \,.\, e^{-qT} \,.\, N_2(a_1,b_1,g) - X \,.\, e^{-rT} \,.\, N_2(a_2,b_2,g)$$

Where

$a_1 = [\ln(S_0/X) + (r - q + \sigma_S^2/2)\,T\,] / (\sigma_S \,.\, T^{0.5})$,

$b_1 = [(\sigma_S^2 - \rho\sigma_R\sigma_S)\,TV - (\sigma_S^2 - \sigma_R^2)\,TV/2 + (PTSR_S - PTSR_R)\,] / (\sigma_V \,.\, TV^{0.5})$,

$a_2 = [\ln(S_0/X) + (r - q - \sigma_S^2/2)\,T\,] / (\sigma_S \,.\, T^{0.5}) = a_1 - \sigma_S \,.\, T^{0.5}$,

$b_2 = b_1 - [(\sigma_S^2 - \rho\sigma_R\sigma_S)TV] / (\sigma_V \,.\, TV^{0.5})$,

$g = [(\sigma_S - \rho\sigma_R)TV^{0.5}] / (\sigma_V \,.\, T^{0.5})$,

$\sigma_V = (\sigma_s^2 - 2\rho\sigma_S\sigma_R + \sigma_R^2)^{0.5}$,

σ_S, σ_R = volatility of S (stock), R (index).

ρ = correlation between S (stock) and R (index).

$PTSR_S$, $PTSR_R$ = Past Cumulative TSR of Stock S and Index R respectively, from grant date until the valuation date. Note that if the valuation is performed on the grant date then $PTSR_S = PTSR_R = 0$.

$N_2(x,y,z)$ = The cumulative bivariate normal distribution representing the joint probability $\{P(X<x) \text{ and } P(Y<y)\}$, where z is the correlation between X and Y.

6.3 Real world projection methodology

The probability of vesting can be derived in a similar manner to arrive at the following closed-form formula:

Prob = $\Pr(\ln(S_{TV}/S_0) > \ln(R_{TV}/R_0))$
= $\Pr(Z < b_2 + [(\mu_s - \mu_R)TV]/[\sigma_V \, . \, TV^{0.5}])$,

where μ_s and μ_R are the instantaneous expected rates of return for S (stock) and R (index) respectively. The other parameters are as per Section 6.2 above.

7. OPTION TYPE 3: RELATIVE TSR HURDLE II

7.1 Description of option

Table 4: A sample design of Option 3

Parameter	Symbol	Description
Hurdle / Target		Cumulative Company TSR must outperform the given Index's TSR for 5 consecutive trading days.
Measurement period		Target must be 'hit' between 3 and 5 years from the date of allocation.
Strike price	X	Average trading price over the 5 trading days preceding initial allocation.
Vesting Period	t between 3 and 5 years	Options vest as soon as the target is hit (within the measurement period).
Period until Exercise	T	Option exercisable at the expiry of 5 years from the initial allocation date.

The main differences between this option and the option described in Section 6 are:

1. The vesting date is not a fixed date (ie. there is a period of time over which the hurdle must be satisfied at least once for vesting to occur).
2. The hurdle requires five consecutive days of cumulative out-performance.

These two features mean that no closed-form formula can be derived, and hence we shall employ simulation methods.

7.2 Valuation methodology

(a) Simulation of measurement period

First of all, we need to simulate the TSR of both the Company and the Index over the initial three-year period. This is performed using the formula[ii]:

$$3\,(r - \sigma_i^2/2) + \sigma\, e_i\, \sqrt{3} \qquad i = 1,2$$

Thereafter, daily incremental returns are added to the cumulative return until the sooner of:

- The company's cumulative TSR ranks ahead of that of the index for 5 consecutive trading days; or
- The end of 5 years (from the outset, ie. a further 2 years) is reached.

Note that since we are dealing with continuous returns, the cumulative return is simply the sum of the three-year return and the incremental daily returns.

The daily incremental returns are calculated using:

$$\Delta t\,(r - \sigma_i^2/2) + \sigma_i\, e_i\, \sqrt{\Delta t}$$

Where Δt = 1 / (number of trading days in a year). This is the length of the average incremental trading period in each year. There are approximately 253 trading days in a year.

(b) Option valuation

For iterations where the target is hit, we must calculate the value of the vested option.

Where this occurs the target will be hit for some value t between 3 and 5 years. Hence the option will have a term of (5 – t) years.

The company's TSR at this point has already been calculated. To obtain the stock price at this time, we must deduct dividends; ie.

$S_t = S_0 . e^{TSR - q\,t}$

where the TSR is the simulated cumulative return at time t, and q is the company's assumed continuous dividend yield. We can then value the option using the Black-Scholes formula:

$C = S\ e^{-qT} . N(d_1) – X\ e^{-rT} . N(d_2)$

where

$S = S_t$, $T=5-t$, $q = q_1$, $\sigma = \sigma_1$
$d_1 = [\ln(S/X) + (r - q_1 + \sigma_1^2/2)T]/(\sigma_1 . T^{0.5})$, $d_2 = d_1 - \sigma_1 . T^{0.5}$

To obtain the present value at the valuation date we need to discount back to time zero at the risk-free rate, ie. Value = $C . e^{-rt}$.

Where the hurdle is not achieved by the end of the 5 year period, a value of zero is registered and included in the running averages.

(c) Simulation routine

The above routine must be performed many thousands of times. After each simulation the running total of the option values should be updated to enable the average value over all simulations to be calculated. This average gives us our estimate of the option value.

7.3 Real world projection methodology

We now turn to the problem of estimating the probability of the target being hit. In this case, we shall also illustrate the calculations for estimating the payoffs at the expiry date of the option.

(a) Target probabilities

Probabilities are estimated using simulation techniques. For each simulation in which the target is reached, a success is recorded. The number of successes, relative to the number of trials, gives us these probabilities.

The required simulation routine can be summarised as follows:

- Three-year returns are projected for both the Company and the Index, using the formula:

  $$3(\mu_i - \sigma^2/2) + \sigma\ e_i \sqrt{3}$$

  where μ_i is the continuous CAPM expected rate of return (per annum), as described in Section 4.

- Thereafter daily returns are projected, and the cumulative performance compared daily. Daily returns are calculated using the formula:

  $$\Delta t\ (\mu_i - \sigma^2/2) + \sigma\ e_i \sqrt{\Delta t}$$

where Δt is defined in the manner described in the option valuation section.

- Each individual simulation ends when either of the following events occur:
  - The Company's cumulative return is greater than the cumulative return for the Index for five consecutive trading days (leading to a successful trial); or
  - The end of five years is reached, without the Company ever reaching the target (leading to an unsuccessful trial).

This simulation process is repeated many thousands of times.

(b) Payoff distributions

For each simulation in (a) above we can further project stock prices and hence option payoffs when the target has been hit.

Define TSR* as the Company's (real world) cumulative TSR where the target has been hit at time t. The stock price at this time will be given by $S_0.e^{TSR^* - q\,t}$ ($=S_t$). We then need to project the stock price a further (5-t) years. The stock price growth rate SGR over this period can be simulated using:

$$SGR = (\mu_1 - \sigma_1^2/2 - q_1) . (5-t) + \sigma_1 e_i (5-t)^{0.5}$$

The stock price at time t=5 is then found from $S_5 = S_t$. exp(SGR), and the option payoff from max(0, S_5 - X). The results from each simulation can then be collated to produce a frequency distribution for payoff levels.

It should be noted that:

- The e_i term used to calculate SGR over the period from 5-t years to 5 years needs to be independent of those used for the period up to time t years.
- Where an unsuccessful trial occurs, a zero payoff needs to be recorded for the purposes of the payoff frequency distribution.

8. OPTION TYPE 4: RELATIVE TSR (PEER GROUP) MEASURE

8.1 Description of option

In this case, the number of options vesting is determined by the relative performance of the executive's company over the first three-year period. This relative performance is with respect to a peer group of stocks chosen from the domestic sharemarket and a number of comparable listed offshore stocks. The performance measure used is TSR. We have also made the assumption that all returns are expressed in domestic currency terms; ie. foreign stock returns are adjusted to allow for exchange rate movements.

Table 5: A sample design of Option 4

Parameter	Symbol	Description
Hurdle / Target		A varying number of options will vest in the executive based on the Company's TSR relative to a peer group of stocks (see below).
Measurement period		Hurdle tested at exactly 3 years from the date of allocation.
Strike price	X	Average trading price over the 5 trading days preceding initial allocation.
Vesting Period	TV	Options vest at exactly three years (at the end of the measurement period).
Period until Exercise	T	Option exercisable at the expiry of 5 years from the initial allocation date.

The percentile rank of the Company's TSR determines the number of options vesting. The number of options vesting at each level of percentile ranking is set out in the table below.

Table 6: Option vesting formula

Percentile Ranking of the Company's TSR within peer group	Percentage of options vesting
0% to 20%	0
20% to 50%	25% increasing linearly to 50%
50% to 80%	50% increasing linearly to 100%
80% to 100%	100%

8.2 Valuation methodology

Given the complexity of this option, we have outlined the simulation process in some detail.

(a) Simulation of measurement period

First of all, we need to simulate the TSRs of both the Company and the peer group stocks. Projections of returns must be handled differently for two distinct groups:

- Domestic stocks; and
- International stocks.

To simulate the three-year total shareholder returns of domestic stocks, we use the now familiar formula:

$3\ (r - \sigma_i^2/2) + \sigma_i\ e_i\ \sqrt{3}$

With foreign stocks we face additional complexity from the consideration of currency. For foreign stocks, the expected returns are expressed in their home currency. Hence some allowance must be made for the expected change (and expected volatility) of the exchange rate and its impact on domestic currency denominated returns.

To take into account the effects of cross-correlations between stocks (both domestic and foreign) and exchange rates, it is suggested that a full projection of individual currencies and foreign stocks be employed. Defining S^f and Q as the foreign stock price and the exchange rate[iii], respectively, we use the following:

1. Foreign Stock TSR (in its home currency)

$\ln[S_T^f\ e^{qT}/S_0^f] = 3\ (r_f - \sigma_i^2/2 - M) + \sigma_i\ e_i\ \sqrt{3}$

where $M = \Sigma_j\ \sigma_{Stock} * \sigma_{Currency} * \alpha(\text{stock, } j^{th} \text{ factor}) * \alpha(\text{currency, } j^{th} \text{ factor})$. The summation is over j=1,2,...,n, where n is the combined number of stocks and currencies being modelled. This quantity is usually small.

2. Growth in Exchange Rate

$\ln[Q_T/Q_0] = 3\ (r - r_f - \sigma_{currency}^2/2) + \sigma_{currency}\ e_{currency}\sqrt{3}$

where r_f is the foreign risk-free rate.

The domestic currency denominated foreign stock TSR (as used for ranking purposes) can then be calculated as:

$\ln[(Q_T S_T^f\ e^{qT})/(Q_0 S_0^f)] = (\ln[S_T^f\ e^{qT}/S_0^f]) + (\ln[Q_T/Q_0])$.

The reasoning underlying the above formulae is non-trivial. Further explanation can be found in Appendix B.

(b) Ranking calculation

The Company's TSR ranking is calculated next. The number of options granted at time t=3 is then determined according to:

- The percentile rankings in the various peer groups; and
- The formula for relating percentile rankings to the number of options (Table 6).

(c) Black-Scholes Calculation

The value of a single option of term 2 years (5 years less the measurement period of 3 years) must now be calculated. Again we use a share price of:

$S_t = S_0 . e^{r^* - q\,t}$ $\qquad$ t = 3

where r* is the Company's simulated cumulative return at time t=3, and q is the Company's assumed continuous dividend yield. We can then value the option using the Black-Scholes formula:

$C = S\,e^{-qT} . N(d_1) – X\,e^{-rT} . N(d_2)$

Where

$S = S_t$, $T=2$, $q = q_1$, $\sigma = \sigma_1$
$d_1 = [\ln(S/X) + (r - q_1 + \sigma_1^2/2)\,T]/(\sigma_1 . T^{0.5})$, $d_2 = d_1 - \sigma_1 . T^{0.5}$.

Discounting back to time zero gives:

Value = $C . e^{-3r}$ x number of options vesting (dependent on the ranking).

Again, this routine must be performed many thousands of times.

9. SOME NUMERICAL EXAMPLES

To illustrate the extent of variation that one might encounter in practice between option plans of various designs, this section provides some sample valuation results, along with a worked example. Section 9.1 contains the worked example, outlining how the simulation process works in practice. Section 9.2 presents some sample valuation results for the different benefit designs outlined in this paper (given a set of assumptions).

9.1 Worked Example: Option Type 3 (Relative TSR Measure)

The following simple example will help to illustrate the simulation process.

Consider the option described in section 7 (the relative TSR performance hurdle, measured between years 3 and 5) under the following assumptions.

Table 7: Option Valuation Variables

Symbol	Description	Value assumed
S_o	Initial company stock price	$20
X	Strike price	$20
q	Dividend yield for company stock	2%
r	Risk free interest rate	6%
σ_s	Volatility of company stock price	20%
σ_i	Volatility of comparison index	16%
$\rho_{i,s}$	Correlation of returns of company and the index	60%

Using these parameters and the methodology outlined above, the first ten simulations of this option yield the results set out in Table 8.

Table 8: Simulation Results

Simulation no	Time hurdle achieved (t)	Simulated TSR at time t	Stock price at time t [1]	Option Value [2]	Discounted option value [3]
1	Not achieved				0.00
2	3.02	-46%	11.91	0.10	0.08
3	Not achieved				0.00
4	Not achieved				0.00
5	3.02	33%	26.16	7.69	6.42
6	3.55	-4%	17.84	1.25	1.01
7	3.02	59%	33.80	14.77	12.32
8	3.22	29%	25.03	6.56	5.41
9	3.08	24%	23.84	5.67	4.72
10	3.02	36%	27.03	8.47	7.07
Average					$3.70

Notes:

1. Calculated as $20 . exp[TSR - 2% . t]

2. Calculated as:

   $\$20 \cdot e^{-2\%(5-t)} \cdot N(d_1) - \$20 \cdot e^{-6\%(5-t)} \cdot N(d_2)$

   where d_1 and d_2 are as per the standard Black-Scholes equation.

3. Discount factor equal to $e^{-6\% \cdot t}$

As per the table, the straight average from ten simulations is equal to $3.70. Using variance reduction techniques (refer Appendix F), after ten simulations this average is $4.75. After 100 simulations this answer converges to $4.30, and after 10,000 simulations to $4.38. This compares to the value for the equivalent vanilla call option (calculated using the Black-Scholes formula) of $4.83.

9.2 Sample Comparison Results

In this section we compare the values found using our methodology for the options described in this paper, given some sample assumptions. These assumptions are set out in the tables below.

Table 9: Summary of Assumptions for Comparison Values

Symbol	Explanation	Value assumed
S_o	Initial company stock price	\$20
X	Strike price	\$20
H	Fixed share price hurdle (type 1)	\$22
q	Dividend yield for company stock	2%
T	Final date for exercise	5 years
r	Risk free interest rate	6%
σ_s	Volatility of company stock price	20%
σ_i	Volatility of comparison index (types 2 and 3)	16%
$\rho_{i,s}$	Correlation of returns of company and the index (types 2 and 3)	60%
Currency	All stocks assumed to be in the domestic currency	n/a
n	Number of stocks being projected (type 4)	6 (including the company's stock)
Correlation matrix	Correlation of returns of modelled companies (type 4)	See Table 10

Table 10: Assumed Correlation Matrix of Option Type 4

Stock No	1	2	3	4	5	6
1	100%	30%	54%	20%	22%	10%
2	30%	100%	30%	12%	5%	31%
3	54%	30%	100%	39%	0%	5%
4	20%	12%	39%	100%	26%	13%
5	22%	5%	0%	26%	100%	25%
6	10%	31%	5%	13%	25%	100%

Table 11: Volatility Assumptions for Option Type 4

Stock no	Volatility assumption
1	20%
2	15%
3	16%
4	17%
5	18%
6	19%

Table 12: Option Valuation Results – 10,000 Simulations (or formula value where possible)

Option type	Description	Value calculated
1	Fixed share price hurdle	$4.10
2	Index TSR hurdle, fixed date comparison	$3.45
3	Index TSR hurdle, 2 year opportunity to beat hurdle	$4.38
4	Peer group TSR hurdle, fixed date comparison	$4.72
Comparison purposes	Black-Scholes formula	$4.83

Caution must be exercised in interpreting these results. Especially one must warn against extrapolating them to specific company circumstances. Individual companies will have their own particular circumstances, and almost all of their executive option plans will have additional features (eg. early exercise features) that are not modelled here - and that may materially impact on value.

This being said, these results are indicative of three important and related features of these plans, namely:

- The application of performance hurdles will, in general, materially reduce the option value; but
- This value will generally be greater than 50% of the Black-Scholes value, even when the performance hurdle is a 50:50 ‘bet’; and
- Different hurdles have materially different impacts on executive option plan values.

10. ASSUMPTIONS

One of the more difficult tasks in setting out valuation and projection problems is deriving appropriate assumptions. Accordingly, in this section we comment on each of the key assumptions.

10.1 Risk-free rate of return

- The risk-free rate should be appropriate to the term of the option.
- The risk-free rate should be expressed as a continuous rate. This may require adjustments to quoted bond yields (often semi-annual compounding rates).
- Generally the most appropriate rate is the zero coupon rate applying to this term. Again this would generally involve an adjustment to available government bond rates.

10.2 CAPM Variables

- We suggest the use of a continuous CAPM expected rate of return. This allows implied stock prices to remain strictly non-negative. Conveniently, cumulative returns can also be obtained by simply adding together the returns of different periods.
- Betas can often be obtained from external data sources, or at least checked against such data.
- It should be checked that the betas, correlation parameters and volatilities assumed are reasonably consistent[iv]. Note that $\beta_i = \rho\sigma_i/\sigma_m$, where σ_i is the volatility of the company's TSR measure, σ_m is the volatility of the market (eg in Australia, the All Ordinaries Accumulation index), and ρ is the coefficient of correlation between the market's and the company's returns.

10.3 Volatility

- This is generally estimated using historic data. A useful starting point is to examine historical data for a period of T years, where T is the term of the option under investigation.
- One should also compare the historical volatility with the volatility implied by option prices quoted in the market for options.
- In examining historic data, special consideration may need to be given to:
  - Outliers in the return data.
  - Special circumstances. Many newly listed companies, for example, have quite high historic volatility relative to the future volatility expected by markets as implied by option prices.

10.4 Foreign Currencies

- In the above analysis we have treated the currency rate Q as the amount of domestic currency per 1 unit of foreign currency. It is important that the data analysed for assumption setting purposes uses this definition. This is required in order to maintain consistency between our methods.
- Note that foreign interest rate assumptions may also be required.

10.5 Dividends

There are generally two methods that can be employed:

- Estimation via historical data; or
- The company may provide this information where, for instance, there are targeted dividend ratios.

It may be worthwhile to investigate the sensitivity of the option value to different dividend policies.

11. LIMITATIONS

There are a number of limitations to the approach put forward in this paper. While we do not consider any of them to be fatal, practitioners do need to be aware of such shortcomings.

11.1 Option Valuation

Many criticisms have been levelled at the Black-Scholes assumptions, in particular:

- That both interest rates and volatility are, in practice, time varying.
- The assumption of the normality of stock returns is somewhat questionable.
- Costless and riskless arbitrage is a useful assumption, but a useful fiction no less.

While acknowledging these issues, we offer the following comments:

- Our starting point in terms of current practice leaves much to be desired. Adopting Black-Scholes analysis is a considerable step forward from some of the rough and ready methods currently employed.
- The determined valuer could readily allow for stochastic volatility and interest rates within the confines of our simulation methodology. Adjusting for the non-existence of riskless and costless arbitrage, we concede, is a little more problematic, but neither is this impossible. Investment banks have resolved these issues for the purposes of their internal pricing models, for options not terribly dissimilar to those considered here.

11.2 CAPM projections

The CAPM is almost certainly more controversial than the Black-Scholes option pricing methodology, and the authors of this paper are certainly not prepared to “die in a ditch” for it. This is a battle more eloquently fought by others, not to mention that there is already a considerable body of literature discussing the pros and cons of the CAPM (too voluminous to cover within the narrow confines of this paper).

To the extent that CAPM is widely accepted, intuitive and useful, when forced with such an existential choice the authors feel inclined to continue with CAPM. Once again, however, it is relatively easy to incorporate other company return models into the framework proposed by this paper.

Those readers with a deeper interest in these issues, and indeed seeking references to the literature, are referred to Appendix A.

12. CONCLUSION

For too long, shareholders and executives have been poorly served by the application of rough and ready valuations of executive options. The authors of this paper firmly believe that the Black-Scholes assumptions, combined with simulation methods and some adjustments where necessary, represents a considerable improvement. Could such an approach form the basis of a valuation standard, allowing for better comparisons between companies, and better information for shareholders and executives?

Perhaps.

APPENDIX A: THEORETICAL AND PRACTICAL CONSIDERATIONS FOR RISK-NEUTRAL VALUATION

The purpose of this appendix is to set out:

- Some of the assumptions underlying our methodology; and
- Our thoughts on some of the common questions that practitioners in this area can expect to encounter.

Sections A.1-A.5 discuss some of the main assumptions of the Black–Scholes equation, and indeed our simulation methodology. It is not meant to be exhaustive, and we refer readers to Musiela and Rutkowski (1997), Baxter and Rennie (1996), and Bingham and Kiesel (1998) for more detailed reviews.

Sections A.6-A.7 consider alternative approaches as well as a number of practical aspects of executive option valuation.

A.1 No arbitrage

The Black–Scholes equation relies on the no–arbitrage principle. That is, if market frictions such as transaction costs, taxes, discontinuous trading, etc are assumed away, it should not be possible to construct a fully hedged portfolio that can risklessly earn more than the risk–free rate. According to the theory, one can:

- Sell an option; and then
- Set up a fully hedged portfolio of stocks and bonds in such a way as to eliminate the risk from the sold option (although as stock prices move over time, this portfolio will have to be rebalanced in order to maintain a fully hedged position).

The next step in the argument is that as the portfolio can be risklessly hedged, the appropriate rate of expected portfolio return, and indeed the discount rate to apply to the portfolio's payoffs, is the risk-free rate.

A.2 Risk–Neutral Growth Rates

There is a further assumption that the rate of return for a stock over time T is normally distributed. Consequently the distribution of the stock price at time T, denoted S_T , is log–normal. This is a useful assumption in that it ensures $S_T \geq 0$. According we require (ignoring dividends):

$S_T = S_o e^{r^*T}$

for some random variable $r^* \sim N(\mu,\sigma)$.

We further require that, as per Section A.1:

$E[S_T] = S_o e^{rT}$

where r is the risk free rate (again, ignoring dividends).

The properties of the log–normal distribution are such that:

$E[S_T] = S_o \exp\{ (\mu + ½\sigma^2) T \}$
ie. $\mu = r - \sigma^2/2$

Hence our formula for modelling stock price returns in the risk-neutral world per unit time is as follows:

$r^* = [(r - \sigma^2/2)T + \sigma T^{0.5} e_i]/T$

Noting that this implies:

$E[r^*] = [(r - \sigma^2/2) T + \sigma T^{0.5} E(e_i)]/T \quad = (r - \sigma^2/2)$

$Var [r^*] = [\sigma^2 T \, Var(e_i)]/T^2 = \sigma^2/T$

Further discussion of this topic can be found in Hull (1998), pages 228 to 231.

A.3 Option Value

Alternatively, using the work of Black and Scholes (1973) we can value options by modelling stock prices using the following stochastic differential equation:

$$dS_t = S_t [r \, dt + \sigma \, dW_t],$$

where W_t represents a standard Brownian motion.

As a consequence, stochastic calculus tells us that:

$$d\ln[S_t] = (r - \sigma^2/2) \, dt + \sigma \, dW_t .$$

By using the theories advanced by Harrison and Kreps (1979), and Harrison and Pliska (1981), we can represent the value of the option as follows:

$$E[e^{-r(T-t)} f(S_T)].$$

Here S_t is modelled using to the above stochastic differential equation, and $f(S_T)$ represents the payoff function. For example, for a call option with strike K:

$$f(S_T) = \max(S_T - K, 0)$$

This expectation is not calculated using real-world probabilities but rather by ‘risk-neutral’ probabilities (called the ‘risk-neutral measure’ or the ‘arbitrage-free measure’)[v].

The formulation of the option valuation problem as an expectation leads quite naturally to a simulation approach to valuation.

A.4 CAPM and Risk-Neutral Valuation

An obvious question is why the expected return of a stock should not be used in the valuation of an option. A related issue is the relationship between risk-neutral option valuation and other asset pricing theories such as the CAPM.

It is of interest that Black and Scholes (1973) presented a derivation of their formula using CAPM assumptions. We refer the interested reader to the excellent monographs by Duffie (1996) and Cochrane (2001) for detailed economic arguments.

A.5 Key Assumptions

The Black-Scholes methodology outlined in Sections A.1 to A.4 made the following key assumptions:

- Investors prefer more to less, which leads to the assumption of no-arbitrage;
- Stock prices can be modelled as a log-normal distribution;
- Interest rates and volatility are constant; and
- The asset being valued is financially traded.

Further, this paper has only considered options exercisable on maturity.

The assumption of non-satiation is not controversial.

The assumption that stock prices can be modelled as log-normal, though, is clearly an approximation. The assumption of constant parameters is linked to the assumption of the log-normality of prices as discussed above. There have been many attempts in the finance and mathematics literature to incorporate more realistic assumptions. These include the stochastic volatility models of Hull and White (1988), and activity time models such as Heyde (1999), amongst many others. As yet no widely accepted alternative model exists. It should be noted that over long timeframes the approximation of log-normality can often be considered reasonable and does not have a large effect on the option values. Executive options often have a long time period (at least at grant) of 5 years or more.

Compared to the log-normal return assumption, the assumption of constant interest rates is more problematic. For long-term options, the potential for interest rate variability can have a material impact on the option's economic value. This area is one deserving of further research. As a guide the model incorporating stochastic interest rates presented in Heath, Jarrow, and Morton (1992) is considered a more than useful starting point.

Many executive options plans allow for the possibility of early exercise – often from the point of vesting onwards. This raises the issue of what value, if any, these early exercise provisions may have. A common approximation is to assume that the option will only be exercised at maturity, ie. the assumption that the option is 'European'. While this feature can have economic value, company-specific circumstances (eg. dividend policy) often make it extremely difficult to value such features in any meaningful way. Further, it is not necessarily the case that such early exercise provisions have any value, especially for low dividend paying companies. On balance, it is probably not unreasonable to assume European exercise features as a baseline for valuation. This issue is one that has rarely been considered in the literature, and constitutes an important area of future research.

A.6 Alternative Approaches

There have been a number of alternative valuation approaches suggested in the literature.

One approach often suggested, both in academia and in practice (as illustrated by the valuation methods used by many companies in the notes to their accounts) is to multiply the Black-Scholes value by an estimate of the probability of meeting the performance hurdle. It has been argued elsewhere (Carrett, Miller, and Roberts (2000)) that this is clearly unsatisfactory.

The key reason for this is that the payoff of an option is highly skewed - as such it will often be the case that an option which fails to meet the performance hurdle will end up being below the exercise price at maturity (and hence worthless). It is likely that for many plans quoting the values of their executive options in this way (eg. 50% of Black-Scholes), the true economic value will often be closer to 90% of the Black-Scholes value. Similar ad-hoc adjustments have also been suggested, especially in the early accounting literature (including the discussion of executive share option values by the Federal Accounting Standard Board of the United States). Again these adjustments are often internally inconsistent and produce answers far from the true economic value.

Yet another related method that has been suggested is to use the Black-Scholes formula but with the risk free rate replaced by the CAPM rate of return. This is equivalent to projecting the returns at μ and discounting by μ. This approach in fact is correct for primary assets (shares) but is erroneous when applied to options. To see this, note that options can be represented as a dynamic portfolio, consisting of varying amounts of shares, that has a time and state dependent risk rate of return. Hence projecting returns and discounting at the rate μ is incorrect.[vi]

It is worthwhile noting that the academic literature on executive option valuation seems to have a pre-occupation with deriving closed-form formulae. To this end many researchers use 'approximations' to the hurdles that can yield closed-form formulae. Closed form formulae are desirable where they yield satisfactory results, and where the alternatives do not justify the additional effort. As discussed earlier, however, many executive option plans have complicated performance hurdles and as such no closed-form formula can be derived[vii]. As solution run-time is not of primary concern we consider it more appropriate to use numerical methods so as to more appropriately consider the effect of performance hurdles.

A.7 Other Company-Specific Practical Considerations

There are a number of considerations for which we do not consider it viable to adjust the value of executive share options. One of these is the right retained by some boards to reset the strike prices post grant date. This has caused some controversy amongst shareholders (a recent example is Microsoft). Such rights are generally subject to significant discretion, the value of which is, in our view, inestimable for most practical purposes. If a 'rule' was defined as to when and if the terms of the options were to be reset, then valuation could be performed as per our methodology, and in certain cases explicit formulae can be found (refer Chance et al (2000)).

Another issue is the possibility of the executive's departure prior to the expiry of the measurement period. In many cases unvested options are forfeited where the executive leaves the firm before vesting occurs. This has led some authors, such as Cuny and Jorion (1995) to suggest reducing option values to allow for the probability of departure. Again this is an area where the Board often has considerable discretion to alter benefit terms. There is also a likely correlation between the value of the option and the tenure of the executive, ie. the executive is more likely to depart where the options are 'underwater', reducing the impact such decrements have on option value. Accordingly, we would generally choose not to model such features.

APPENDIX B: RISK NEUTRAL STOCHASTIC PROCESSES

The appendix sets out the form of some of the stochastic processes that underlie risk-neutral valuation. As such these will be the stochastic processes driving our risk-neutral simulation algorithms.

By necessity this appendix is highly mathematical. For further references on the mathematics used refer to the monographs by Baxter and Rennie (1996) and Musiela and Rutkowski (1997).

B.1 Domestic Share Price

We will assume that the 'real world' share price process S_t follows geometric Brownian motion, and is governed by the probability measure P^+. Its dynamics are given by the following stochastic differential equation:

$$d[S_t] = S_t [\mu \, dt + \sigma_S \, dW^+_t],$$

where W^+_t is a P^+ - Brownian motion.

In the above equation μ is the expected (instantaneous) return on the share price. We will also define q as the (instantaneous equivalent) rate of dividend payment. Alternatively a discrete dividend 'jump' can be incorporated.

One example of $\mu + q$ would be the return as defined by an asset pricing model such as the CAPM or the APT.

Define $B_t = Be^{rt}$ as the accumulation of \$1 from time 0 to time t at a rate r per annum:

$$dB_t = r \, B_t \, dt.$$

The process of concern is $S_t e^{qt} / B_t$, which represents the value of the stock price, accumulated with reinvested dividends, and discounted at the risk free rate:

$$d(S_t \, e^{qt}/B_t) = (S_t / B_t) \, . \, [(\mu + q - r) \, dt + (\sigma_S) \, dW^*_t].$$

From the theories of Harrison and Kreps (1979), Harrison and Pliska (1981) we can value options by enforcing that any tradeable asset discounted by the savings account B_t must be a martingale under the equivalent martingale measure P (we will also call this the domestic martingale measure[viii] in the following sections). This implies that for option valuation purposes we can treat the dynamics of the share price as follows:

$$\mu = r - q$$

Consequently:

$$dS_t = S_t [(r - q) \, dt + \sigma_S \, dW_t],$$

where W_t is a P - Brownian motion.

It is often easier to work with returns, defined as $\ln(S_t / S_0)$. Stochastic calculus tells us that:

$$d[\ln(S_t)] = (r - q - \sigma_S^2/2) \, dt + (\sigma_S) \, dW_t .$$

B.2 Domestic TSR

The total shareholder return (TSR) of a stock is approximately the sum of the share price growth and reinvested dividends. Hence we can model this as:

$$d[\ln(S_t)] = (r - \sigma_S^2/2) \, dt + (\sigma_S) \, dW_t.$$

B.3 Domestic Index TSR

A good approximation is to treat an index (R) as similar to any other stock, ie.

$d[\ln(R_t)] = (r - \sigma_R{}^2/2)\, dt + (\sigma_R)\, dW_t$.

Note that this assumption implies that the index's returns are log-normally distributed. This is slightly inconsistent with the log-normality assumption for stocks. This is because the index is usually constructed using a weighted average of individual stock returns, but the sum of several log-normal distributions is not log-normal. Fortunately this difference will only be material if the number of stocks underlying the index is very small. For indices composed of a large number of stocks (eg 30 or more) this should be a reasonably good approximation.

B.4 Foreign Currency

The modelling of foreign currencies is more complicated. Define Q_t as the exchange rate, ie. the amount of local currency per 1 unit of the foreign currency. We will assume that the rate is log-normally distributed under the real world measure P^+:

$dQ_t = Q_t\, [\, \mu\, dt + \sigma_Q\, dW^+{}_t\,]$.

Note that this is not a traded asset. A traded asset, however, can be represented by $Q_t\, B^f{}_t$ where $B^f{}_t$ is the foreign savings account:

$B^f{}_t = e^{f\,t}$, $B^f{}_0 = 1$,

where f is the foreign risk free rate.

Hence we have:

$d\, [B^f{}_t\, Q_t / B_t] = B^f{}_t\, Q_t / B_t\, [\, (\mu + f - r)\, dt + (\sigma_Q)\, dW^+{}_t\,]$

This must be a martingale under the domestic martingale measure P. Hence we require that:

$\mu = r – f$.

(Note that this is a consequence of the covered interest rate parity theory of international finance). Hence under the domestic martingale measure P:

$dQ_t = Q_t\, [\, (r - f)\, dt + (\sigma_Q)\, dW_t\,]$

$d\ln[Q_t] = (r – f - \sigma_Q{}^2 / 2)\, dt + \sigma_Q\, dW_t$.

B.5 Foreign Share Price (in foreign currency)

We will be taking the perspective of a foreign investor in this section. We will refer to the arbitrage-free measure for this foreign investor as the foreign martingale measure, P^f. Using arguments analogous to that used in section B.1, we know that, to the foreign investor, the foreign stock price discounted at the foreign risk free rate must be a martingale under P^f:

$d[\ln(S^f{}_t)] = (f - \sigma_{Sf}{}^2/2)\, dt + \sigma_{Sf}\, dW^f{}_t$.

Note, however, that we cannot use these dynamics to value our option because they relate to the foreign martingale measure P^f. To value the option we require the dynamics under the domestic martingale measure P.

B.6 Foreign Share Price (in domestic currency)

In section B.5 we have derived the dynamics of $S^f{}_t$ from the foreign investor's perspective (under P^f). The option valuation problems we are faced with, however, relate to the domestic investor. Hence the aim of this section is to derive the dynamics of $S^f{}_t$ under the domestic martingale measure P.

Note that since we are working with more than one risky asset, we have to define W_t as an n-dimensional Brownian motion. This is required to ensure non-perfect correlation between the different asset classes. Furthermore define $(\sigma_Q)\ dW_t$ as the sum (over i=1 to n) of the i^{th} component of σ_Q multiplied by the i^{th} component of dW_t.

To determine the P-dynamics of S^f_t we first need the dynamics of $(1/\ Q_t)$ under P^f. Using stochastic calculus we find that:

$$d[1/Q_t] = [1/Q_t]\ [\ (f - r + \sigma_Q^2)\ dt\ \ - (\sigma_Q)\ dW_t\ ]$$

(This equation is a demonstration of Siegel's paradox as explained in Hull (1998), pages 299-301).

Using the same reasoning as that of section B.4 (but looking at this as a foreign instead of a domestic investor), we know that a tradable asset in the foreign market will be $B_t\ /\ (Q_t)$. The dynamics of this, when discounted at the foreign risk free rate, are as follows:

$$\begin{aligned} d[B_t / (B^f_t\ Q_t)] &= [1/Q_t]\ [\ \sigma_Q^2\ dt\ \ - \sigma_Q\ dW_t\ ] \\ &= -\sigma_Q\ [1/Q_t]\ [-\sigma_Q\ dt\ +\ dW_t\ ]. \end{aligned}$$

But since $B_t\ /\ (B^f_t\ Q_t)$ is the discounted value of a tradable asset in the foreign market, we know that this must also be a martingale under P^f, ie:

$$d\ [B_t / (B^f_t\ Q_t)] = -(\sigma_Q)\ [1/Q_t]\ dW^f_t.$$

On equating the above formulae we have the relationship between W^f_t and W_t:

$$W^f_t\ = W_t\ - \sigma_Q\ t.$$

From this we can derive that the foreign stock price dynamics of S^f_t under the domestic martingale measure P as follows:

$$\begin{aligned} d[\ln(S^f_t)] &= (f - \sigma_{Sf}^2/2\ )\ dt\ + (\sigma_{Sf})\ (dW^f_t - \sigma_Q\ dt) \\ &= (f - \sigma_{Sf}^2/2)\ dt\ + (\sigma_{Sf})\ (dW^f_t - \sigma_Q\ dt) \\ &= (f - \sigma_{Sf}^2/2 - \sigma_Q .\ \sigma_{Sf}\ )\ dt\ + (\sigma_{Sf})\ (dW_t). \end{aligned}$$

Note that $\sigma_Q .\ \sigma_{Sf}$ is denoted in matrix form. In non-matrix notation this is equivalent to:

$$\Sigma_j\ \sigma_{Stock} * \sigma_{Currency} * \alpha(\text{stock}, j^{th}\ \text{factor}) * \alpha(\text{currency}, j^{th}\ \text{factor})$$

This summation[ix] is over j=1,2,...,n where n is the combined number of stocks and currencies being modelled. The α factors are determined by the Cholesky decomposition as outlined in Appendix C.

The dynamics of both the foreign stock price and the exchange rates have now been derived. Using the two results above we can then model the foreign share price, in domestic currency, using:

$$d[\ln(Q_t\ S^f_t)] = d[\ln(Q_t)] + d[\ln(S^f_t)]$$

where $d[\ln(Q_t)]$ is of the form shown in Section B.4.

APPENDIX C: CHOLESKY DECOMPOSITION

The Cholesky decomposition is a means to transform independent random variables into random variables with a given correlation structure, but without changing the means and variances of the underlying variables.

The problem can be defined as follows. Begin with a series of independent random variables e_i, i = 1,2,…..,n.

We require a linear transformation of these variables such that:

$e_i^* = \Sigma \alpha_{ik} e_k$,

where e_i^* has the same mean and variance as e_i for each i.

The problem is most tractable in matrix form.

Define
$e = (e_1, e_2, \ldots, e_n)^T$, $e^* = (e_1^*, e_2^*, \ldots, e_n^*)^T$

$$\alpha = \begin{matrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2n} \\ .. & .. & .. & \cdots & .. \\ .. & .. & .. & \cdots & .. \\ \alpha_{n1} & \alpha_{n2} & \alpha_{n3} & \cdots & \alpha_{nn} \end{matrix}$$

$$\Sigma = \begin{matrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 & .. \\ \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & .. & .. \\ .. & .. & .. & .. \\ .. & .. & \rho_{n,n-1}\sigma_{n-1}\sigma_n & \sigma_n^2 \end{matrix}$$

In matrix form, we require a matrix α such that:

$e^* = \alpha e$.

This matrix must satisfy:

$$\alpha\alpha^T = \Sigma.$$

By solving for α we can transform a vector of uncorrelated normal variables into a set of correlated variables with variance-covariance structure Σ.

Note that the Cholesky decomposition is not the only method that can be used. It is, however, a commonly used and efficient algorithm for solving this problem. Other methods such as eigenvalue methods can be used to arrive at another solution.

The interested reader is referred to Wilmott (1998) for an algorithm capable of deriving the necessary matrix.

APPENDIX D: NUMERICAL ALGORITHMS FOR THE (UNIVARIATE AND BIVARIATE) CUMULATIVE NORMAL DISTRIBUTION FUNCTION

This appendix discusses the functions used to solve for the cumulative probability (and inverse) of the normal distribution.

D.1 Cumulative density function for a univariate normal

One means of calculating Prob $(Z < X)$ where $Z \sim N(0,1)$ is to use built-in spreadsheet functions. When performing a large number of complex simulations, calling spreadsheet functions in this way is not computationally efficient. A simple algorithm to generate this function can be found in Hull (1998), Chapter 11.

D.2 Cumulative density function for a bivariate normal

This function is required for valuation of options with a hurdle tested at a single date. Again an algorithm to calculate this function can be found in Hull (1998), Chapter 11.

D.3 Inverse Cumulative density function for a univariate normal

This function is important as it is used to generate normally distributed random variates using random numbers generated uniformly along the interval [0,1] as inputs. Again it is typically more computationally efficient to use embedded program code to perform this task than to call, for instance, an in-built spreadsheet function.

A quick and simple algorithm has been described in an article by Moro (1985), ‘The Full Monte’ in Risk Magazine. We refer the reader to this article for details.

APPENDIX E: RANDOM NUMBER GENERATION

This appendix briefly describes several issues relating to random number generation. It should be noted that the concepts in this Appendix are closely related to the concepts of variance reduction outlined in Appendix F.

More detailed discussion on the material in this and the following appendix can be found in many excellent sources, for example Press et al (1995), and Gentle (1998).

E.1 Visual Basic RND Function

The Visual Basic function RND generates a pseudo random number uniformly distributed on the range (0,1). As this is an inbuilt Visual Basic function, users often choose it. This function is not generally suitable for our purposes for the reasons set out below.

One point to note is that before a sequence is generated (using RND) the following statement should be used:

RANDOMIZE(x)

Where x is any integer less then 2^{24}-1. This statement determines the seed that is used to generate the set of random numbers. For a given x the same sequence of random numbers will be generated. This is useful where one is checking models – the differences between various runs will not be due to the use of a different sequence of random numbers.

Note, however, that the same sequence of random numbers will eventually repeat after at most 2^{24}-1 numbers are generated (and often within much shorter time frames, depending on the seed x chosen). Accordingly this method is not recommended if a large quantity of random numbers is required.

E.2 Alternative methods

Most pseudo random number generation algorithms use an equation of the form:

$$I_{j+1} = (a * I_j + c) \bmod m$$

where I_{j+1} is the j+1 th pseudo random number, and a, c and m are constants.

For most purposes and sets of constants a, c and m this method is adequate. Tests (see also section E.3) have shown that such pseudo random numbers will appear reasonably random. Due to the 'mod m' condition, however, as well as the fact that each number depends on the previous number, such sequences can only generate m random numbers before the sequence repeats itself. The quantum of numbers that can be generated is called the period of the generator. Sometimes the period is less than m, depending on the choice of the constants a and c.

The RND function in Visual Basic, for instance, has a maximum period of 2^{24} or roughly 16 million. Furthermore, the effectiveness of the sequence depends on the initial seed (as defined by the RANDOMIZE statement).

For our purposes we sometimes need more than 50 million pseudo random numbers to ensure a stable answer. In these instances the RND function is clearly inadequate – hence the need for a different generator.

Another deficiency occurs with respect to multi-dimensional problems (e.g. generating random variables for several stocks) – as is the case for most executive options with performance hurdles. The RND generator performs poorly in this context also.

E.3 The PMBD method

An excellent comparison of different pseudo random number generation techniques can be found in the book 'Numerical Recipes in C' by Press et al (1992). Based on their examination of speed, randomness and period they recommend the algorithm developed by Parks and Miller, with adjustment by a Bays-Durham shuffle, which we shall refer to as the Parks-Miller-Bays-Durham ('PMBD') method. We have adopted this as our standard algorithm to generate pseudo random numbers. This method is explained further below.

Parks and Miller noted that the multiplicative congruential generator

$$I_{j+1} = (a * I_j) \bmod m$$

(ie. c=0), is often good enough when appropriate values of a and m are used. They suggest the following (called the 'minimal standard'):

$$a=7^5, \quad m=2^{31}-1$$

With these parameters the period of the generator is around 2 billion.

The problem of multi-dimensionality stems from a low-order serial correlation inherent in these generators. To get around this problem, Press et al further recommend the use of the Bays-Durham shuffle in conjunction with the Parks and Miller method. In short, this method generates the pseudo random numbers as above, but the random numbers are 'shuffled' amongst a group of 32 numbers. Each run draws one number from the 32 numbers at random, and replaces it with another number.

In short, for option valuation problems the Visual Basic RND function is often not sufficient. When generating multi-dimensional pseudo random numbers the PMBD method is a considerably better solution.

E.4 Testing the 'Degree of Randomness'

Whenever a set of pseudo random numbers is used it is often a good idea to check whether the numbers generated appear 'random'. The simplest way to do this is to check the fit of the U(0,1) variables using a chi-squared test. For example, one may divide the interval (0,1) into, say, 100 equal sub-intervals, and compute the chi-squared statistic:

$$\chi^2 = \Sigma_{i=1 \text{ to } 100} [Observed_i - Expected_i]^2 / Expected_i$$

Where:

$Observed_i$ = Number of samples observed in the interval [(i-1)/100, i/100]

$Expected_i$ = Number of samples expected in the interval [(i-1)/100, i/100]

= (1/100)*number of generated numbers

This χ^2 statistic will have 99 degrees of freedom. Standard statistical tests can then be used on this statistic to check whether the hypothesis that the numbers are not random enough is rejected (ie. $P(X>\chi^2) < 0.05$).

These tests are important as a poor sequence of pseudo random numbers can bias the simulation result and may cause convergence of the result to an incorrect answer as the number of simulations increase.

Note that when the problem is 2-dimensional it is recommended that a chi-squared test be used on the {(0,1) , (0,1) } unit square. This is because serial correlations can cause a bias on multi-dimensional random variables even if the series is adequate for 1-dimensional problems. A classical example would be the multiplicative congruential generator (including the Parks-Miller minimum standard) – a 2-dimensional set of points generated from successive points of the sequence will fail many statistical tests of fit. (Hence the use of the Bays-Durham shuffle).

E.5 Recommended Approach

For well-defined problems where a formula may be derived it is suggested that the formula be used to provide the solution of the option valuation problems. Conversely, even if a closed form formula does exist, it is useful to investigate the value via simulation, as this provides a check on the closed-form formulae.

APPENDIX F: Variance Reduction methods for simulation

While the crude Monte Carlo simulations illustrated in Section 9 are quite easy to program and use, the standard error of our estimate (of the option value) can be unacceptably large for quite high numbers of simulations. Furthermore, as shown in Appendix E, problems with generating sufficient random numbers will sometimes limit the number of simulations that can be made. To circumvent these problems several 'variance reduction' procedures can be used. While there are many different such methods available we will concentrate on three simple methods that can be readily implemented: the methods of antithetic variables, control variates, and conditional Monte Carlo.

A very readable and thorough discussion of variance reduction techniques as used in mathematical finance and option pricing can be found in Boyle, Broadie and Glasserman (1997).

F.1 Antithetic Variables

The use of antithetic variables is a significant aid to rapid convergence. The basic idea is as follows: For a sequence of N(0,1) random numbers e_1, e_2, .., e_n, the sequence $-e_1$, $-e_2$, .., $-e_n$ is also a legitimate N(0,1) sequence. If we use a sequence comprising: e_1, e_2, .., e_n, $-e_1$, $-e_2$, .., $-e_n$ then we can in effect have 2n random numbers – at a lower cost (in terms of time) since only half the random numbers need to be generated. Furthermore, the variance of the estimate will be lowered as the correlation between the two series is negative. Note that the correlation structure between the e_i is preserved.

This method is very easy to implement as it only requires several additional lines of code.

Confidence intervals should be constructed using the average of the original and the antithetic paths, ie.

$$f_{ave} \quad = \quad 0.5 * (f_e + f_{-e})$$

(This is because f_e and f_{-e} are not independent).

F.2 Control Variates

Control variates involve relating the value of the option under consideration to the value of a related option for which a closed-form formula exists.

Consider the estimation of a function (e.g. an option value) f(S). If we have another function g(S) that is:

- correlated with f(S); and
- has a closed-form formula;

we can obtain an improved estimate of f(S) by using the following:

$$f_{IMPROVED}(S) = f_{ESTIMATED}(S) + [g_{ACTUAL}(S) - g_{ESTIMATED}(S)]$$

The functions $f_{ESTIMATED}(S)$ and $g_{ESTIMATED}(S)$ should be generated using the same sequence of (pseudo) random numbers S.

Note that by using the 'control variate' g(S) we will not be inducing a bias in the estimate of f(S). This can be seen by the following:

$$E[f_{IMPROVED}(S)] = E[f_{ESTIMATED}(S)] + E[g_{ACTUAL}(S) - g_{ESTIMATED}(S)]$$

The function $E[g_{ACTUAL}(S) - g_{ESTIMATED}(S)]$ will have an expected value of 0. The use of antithetic variables and large numbers of simulations should ensure that the net effect of the control variate will be small. The use of the control variate can, however, significantly assist faster convergence for both smaller numbers of simulations and complex payoffs.

F.3 Conditional Monte Carlo

Conditional Monte Carlo methods help to reduce variance due to the fact that:

$$\mathrm{Var}(X) \geq \mathrm{Var}(E(X \mid Y))$$

In short, the method tries to ensure that we use simulation as little as possible and rely on closed-form integration as much as possible.

For example, we can value the executive option using the Black-Scholes formula as soon as the option vests, rather than simulating the performance of the stock price until maturity (ie {Y=Black-Scholes price} in the above equation).

This method is simple to implement, and is quite effective in increasing efficiency and convergence of the Monte Carlo solution.

APPENDIX G: SELECTED REFERENCES

Texts and papers that are relatively accessible and deal with the ideas highlighted in this paper include:

(a) Research on Executive Options Valuation

Acharya, V., John, K, & Sundaram, R., 2000, 'On the optimality of resetting executive stock options', *Journal of Financial Economics*, Volume 57, Issue 1.

Brenner, M., Sundaram, R. & Yermack, D., 2000, 'Altering the terms of executive stock options', *Journal of Financial Economics*, Volume 57, Issue 1.

Carpenter, J. & Yermack, D., 1998, (eds), *Executive compensation and shareholder value: theory and evidence,* Kluwer Academic.

Chance, D., Kumar, R., & Todd, R. 2000, 'The 'repricing' of executive stock options', *Journal of Financial Economics*, Volume 57, Issue 1.

Carrett, P., Miller, R. & Roberts, Z. 2000, 'Executive Options: How Should They Be Valued?', *Tillinghast Briefings for the Financial Services Industry*, No 56, February Edition.

Cuny, C. & Jorion, P. 1995, 'Valuing Executive Options with Endogeneous Departure', *Journal of Accounting and Economics*, 20.

Dayananda, P. 2000, 'Executive Stock Options: Valuation Based on Targeted Performance', *North American Actuarial Journal*, Vol 4, #1, January.

Heath, D., Jarrow, R. & Morton, A. 1992 'Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation'. *Econometrica* 60, 77-106

Hemmer, T., Matsunaga, S. & Shevlin, T. 1996, 'The influence of risk diversification on the early exercise of employee stock options by executive officers', *Journal of Accounting and Economics 21*, pp. 45–68

Huddart, S. 1998, 'Patterns of Stock Option Exercises in the United States', in Carpenter, J., & Yermack, D 1998 (eds).

Huddart, S. & Lang, M. 1996, 'Employee stock option exercises: an empirical analysis'*, Journal of Accounting and Economics* 21, pp. 5–43

Johnson, S. & Tian, Y. 2000, 'The value and incentive effects of non-traditional executive stock option plans', *Journal of Financial Economics*, Volume 57, Issue 1.

Johnson, S. & Tian, Y. 2000, 'Indexed executive stock options', *Journal of Financial Economics*, Volume 57, Issue 1.

Statement of Financial Accounting Standards No. 123, 1995, 'Accounting for Stock-Based Compensation', Financial Accounting Standards Board.

(b) General Reading

Nelson, H.G., 1996, 'Petrol, Bait, Ammo and Ice', Pant.

(c) Derivatives Theory

Baxter, M. & Rennie, A. 1996, 'Financial Calculus: An Introduction to Derivative Pricing', Cambridge University Press.

Bingham, N. & Kiesel, R. 1998, 'Risk Neutral Valuation', Springer.

Hull, J. 1998, 'Options, Futures and Other Derivatives', International Edition, Prentice Hall.

Wilmott, P. 1998, 'Derivatives the Theory and Practice of Financial Engineering', John Wiley & Sons.

(d) Monte Carlo Simulation and Random Number Generation

Clewlow, L. & Strickland, C. 1998, 'Implementing Derivatives Models', John Wiley & Sons.

Gentle, J. 1998, 'Random Number Generation and Monte Carlo Methods', Springer.

Hossack, I., Pollard J. & Zehnwirth, B. 1983, 'Introductory Statistics with Applications to General Insurance', Cambridge University Press.

Press, W.H., Teukolsky, S.A., Vetterling, W.T. & Flannery, B.P. 1995, 'Numerical Recipes in C: The Art of Scientific Computing', Second Edition, Cambridge University Press, 1992.

Seila, A., 1998, 'Monte Carlo Simulation for Actuarial Problems', *Record of the Society of Actuaries*, 21, #3A, 327-349.

More technical papers and texts on the ideas and methods used in this paper include:

Black, F., & Scholes, M. 1973, 'The Pricing of Options and Corporate Liabilities', *Journal of the Political Economy*, 81, 637-659

Boyle, P. 1977, 'Options: A Monte Carlo Approach', *Journal of Financial Economics*, 4, 323-338.

Boyle, P., 1996, 'Valuation of Exotic Options Using the Monte Carlo Method', in Nelken, I. (ed), *The Handbook of Exotic Options: Instruments, Analysis and Applications*, Irwin Professional.

Boyle, P., Broadie, M. & Glasserman P. 1997, 'Monte Carlo Methods for Security Pricing', *Journal of Economic Dynamics and Control*, 21, 1267-1321.

Cochrane, J. 2001, 'Asset Pricing', Princeton University Press.

Duffie, D. 1996, 'Dynamic Asset Pricing Theory', 2nd Edition, Princeton University Press.

Harrison, J. & Kreps D. 1979. 'Martingales and arbitrage in multi-period securities markets', *Journal of Economic Theory*, 20, 381-408.

Harrison, J. & Pliska, S. 1981, 'Martingales and stochastic integrals in the theory of continuous trading', *Stochastic Processes and their Applications*, 11, 218-260

Heyde, C. C. 1999, 'A risky asset model with strong dependence through fractal activity time', *Journal of Applied Probability*.

Hull, J. & White A. 1988, 'The Use of the Control Variate Technique in Option Pricing', *Journal of Financial and Quantitative Analysis*, 23, 237-251

Lehoczky, J. 1997, 'Simulation Methods for Option Pricing', in Rogers, L. & Talay D., (eds), *Numerical Methods in Finance* (Publications of the Newton Institute), Cambridge University Press.

Moro, B. 1985, 'The Full Monte', *RISK*, February 1985, pp 57-58,

Musiela, M. & Rutkowski M. 1997, 'Martingale Methods in Financial Modelling', *Applications of Mathematics*, 36, Springer Verlag.

ENDNOTES

[i] Refer to Hull (1998), pages 210-217.

[ii] Where the starting date for the cumulative return is different to the valuation date, appropriate adjustments should obviously be made.

[iii] We will define Q as that rate which converts foreign currency into domestic currency. For example, assuming the domestic currency is AUD and the foreign currency is USD, Q is the amount of AUD per USD$1.

[iv] For example, some data providers use a Bayesian approach to calculate betas, whereas we may use a different approach to calculate the volatilities. Any inconsistencies should be relatively immaterial and capable of rationalisation.

[v] Note we are not assuming that investors are risk-neutral.

[vi] Actually this idea can be correctly implemented by using a different rate of return, but the share's risk rate of return μ is clearly not the risk rate of return for the option.

[vii] In particular a 'closed form' formula may include a high dimension multivariate normal distribution function that requires numerical techniques anyway.

[viii] "Domestic" variables relate to the perspectives of domestic investors.

[ix] For the simple case when W is 2-dimensional ie the uncertainties relate only to the foreign stock and the exchange rate, the (non-matrix) form of $\sigma_Q . \sigma_{Sf}$ will be $\rho\, \sigma_Q\, \sigma_{Sf}$.